COMPRESSIVE POWER SPECTRAL DENSITY ESTIMATION

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ABSTRACT

In this paper, we consider power spectral density estimation of bandlimited, wide-sense stationary signals from sub-Nyquist sampled data. This problem has recently received attention from within the emerging field of cognitive radio for example, and solutions have been proposed that use ideas from compressed sensing and the theory of digital alias-free signal processing. Here we develop a compressed sensing based technique that employs multi-coset sampling and produces multi-resolution power spectral estimates at arbitrarily low average sampling rates. The technique applies to spectrally sparse and nonsparse signals alike, but we show that when the wide-sense stationary signal is spectrally sparse, compressed sensing is able to enhance the estimator. The estimator does not require signal reconstruction and can be directly obtained from a straightforward application of nonnegative least squares.

Index Terms— power spectral density estimation, multi-coset sampling, compressed sensing, nonnegative least squares

1. INTRODUCTION

A number of compressed sensing (CS) structures have been proposed for sub-Nyquist sampling of continuous-time signals [1–4].[1] All of these systems extend CS theory and serve as viable mechanisms to "compressively" acquire these types of signals. Concurrently, research has focused on finding new ways to directly process CS measurements (once acquired) to infer or extract information without having to reconstruct the original signal or its Nyquist samples. For example, in [5] Davenport et al. consider the fundamental tasks of detecting, classifying, and estimating deterministic signals in Gaussian noise from within the "compressed" domain. Together, these two lines of research attempt to realize one of the central goals of CS theory—to directly *acquire and process* the salient information in a signal (where possible) without intermediary data compression or signal reconstruction steps and in turn, increase the efficiency and the capability of current signal acquistion systems.

In this paper, we further develop and blend these ideas and propose a novel "compressive" power spectral density (PSD) estimator for sub-Nyquist sampled wide-sense stationary (WSS) random signals $x(t)$. The method utilizes the multi-coset (MC) sampler, originally proposed by Feng and Bresler [1], and relies on the fact that the Fourier transform of a WSS signal is a (nonstationary) white noise process [6] to form a linear system of equations whose solutions represent the power in various spectral bands of $x(t)$. The estimates are therefore *finite resolution approximations* to the true PSD that can be computed at *arbitrarily low sampling rates*. We also show that the method's solutions fall within two categories depending on whether one assumes $x(t)$ is spectrally sparse. (The notion of spectral sparsity is defined in Section 2.) A surprising discovery is that the proposed PSD estimator can take the form of a nonnegative least squares estimate regardless of the sparsity of $x(t)$. Because of the structure of the problem, we can recover a sparse solution (when $x(t)$ is assumed sparse) by simply finding a least squares solution instead of resorting to more complicated sparse approximation techniques, e.g. ℓ_1 minimization.

CS based spectral estimation has recently gained attention and found application in cognitive radio systems, e.g., [7], where they are used to monitor a crowded radio spectrum searching for underutilized bandwidth. The proposed estimator also shares many aspects with digital alias-free signal processing (DASP) [8, 9], and in particular, can be viewed as a type of DASP spectral estimator. The link with DASP is briefly examined in Sections 2 and 3.

We begin by giving a brief overview of alias-free and multi-coset sampling in Section 2. Then, in Section 3, we present the PSD estimator and provide two numerical examples in Section 4. We conclude in Section 5.

2. ALIAS-FREE AND MULTI-COSET SAMPLING

Alias-free sampling. Alias-free sampling is a type of non-uniform sampling technique where the set of sampling instants $\{t_n\}$ satisfies certain properties such that the spectrum of the resulting discrete WSS process avoids aliasing and/or yields consistent PSD estimates [8,9]. Typically, the set $\{t_n\}$ is a stochastic point process (e.g., a Poisson point process) that is statistically independent of the signal. The advantage of such methods, in comparison to uniform sampling, lies in the fact that aliasing can be avoided even when the average sampling rate falls below the Nyquist rate. In fact, consistent alias-free PSD estimators are possible with arbitrarily low sampling rates (although such estimates may require long acquistion times) [8].

Alias-free PSD estimators often take a form similar to the standard periodogram defined for uniform sampling. For example, letting $x(t)$ be a real WSS stochastic process with zero mean and PSD $P_x(\omega)$, the following estimator $\widehat{P}_x(\omega)$ is known to be a consistent estimate of $P_x(\omega)$ when the sampling set $\{t_n\}$ is a Poisson point process [8]:

$$\widehat{P}_x(\omega) = \frac{1}{\pi \lambda N} \sum_{n=1}^{N-1} \sum_{k=1}^{N-n} x(t_k) x(t_{k+n}) \quad w_N(t_{k+n} - t_k) \cos(\omega(t_{k+n} - t_k)),$$

where λ denotes the rate parameter of the Poisson process and the window function $w_N(t)$ controls the variance of the spectral esti-

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[1] We include MC sampling in this list because of its similiarities to the more recent CS techniques, despite the fact that it significantly predates CS.

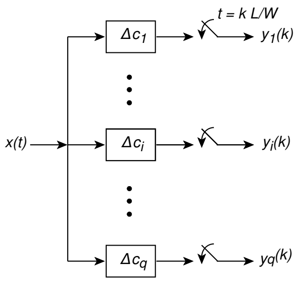

Fig. 1. Multi-coset sampler implemented as a multi-channel system.

mate. The proposed method outlined below accomplishes the same feat as alias-free PSD estimators in that it produces PSD estimators at arbitrarily low sampling rates using MC sampling and CS recovery techniques.

Multi-coset sampling. Multi-coset sampling is a periodic nonuniform sub-Nyquist sampling technique for acquiring sparse multiband signals [1]. A *multiband signal* $x(t)$ is a bandlimited, continuous-time, squared integrable signal that has all (or most) of its energy concentrated in one or more disjoint frequency bands (of positive Lebesgue measure). A *spectrally sparse* multiband signal is a multiband signal whose spectral measure is small relative to the overall signal bandwidth. If, for instance, all the active bands have equal bandwidth B Hz and the signal is composed of K disjoint frequency bands, then a sparse multiband signal is one satisfying $KB \ll W$, where $W/2$ is the bandwidth of $x(t)$ and W is therefore the Nyquist frequency.

For the remainder of this section, let $x(t)$ denote a deterministic sparse multiband signal. For a fixed time interval T that is less than or equal to the Nyquist period and for a suitable positive integer L, MC samplers sample $x(t)$ at the time instants $t = (kL + c_i)T$ for $1 \leq i \leq q$, $k = 0, 1, \ldots$. The time offsets c_i are distinct, positive real numbers less than L and are known collectively as the multi-coset *sampling pattern*. The system thus collects $q \leq L$ samples in LT seconds, or equivalently, exhibits an average sampling rate of q/LT Hz. Here we set T equal to the Nyquist period $T = 1/W$, thereby referencing the system's sampling rate to the Nyquist rate. Multi-coset samplers are parameterized by q, L, and $\{c_i\}$, and the system design depends on conditioning them properly to ensure successful recovery of $x(t)$ from the output samples. MC samplers are most easily implemented as multichannel systems where channel i shifts $x(t)$ by c_i/W seconds and then samples *uniformly* at W/L Hz (see Figure 1).

Using standard properties of the Fourier transform and the discrete-time Fourier transform [10], we obtain a frequency domain description of the output sequences $y_i(k) = x(kL/W + c_i/W)$,

$$Y_i(e^{j\omega\frac{L}{W}}) = \frac{W}{L} \sum_{m=-\left\lfloor \frac{L}{2}(1-\frac{\omega}{\pi W})\right\rfloor+1}^{\left\lfloor \frac{L}{2}(\frac{\omega}{\pi W}+1)\right\rfloor} X(\omega - 2\pi\tfrac{W}{L}m)e^{j\frac{c_i}{W}(\omega-2\pi\frac{W}{L}m)},$$

where $Y_i(e^{j\omega\frac{L}{W}})$ denotes the discrete-time Fourier transform of the sequence $y_i(k)$, $X(j\omega)$ denotes the Fourier transform of $x(t)$, and $\lfloor\cdot\rfloor$ denotes the floor function. The summation limits are finite for a given ω because $x(t)$ is assumed bandlimited. Because $Y_i(e^{j\omega\frac{L}{W}})$ is periodic with period $2\pi W/L$, we can, without loss of information, restrict $Y_i(e^{j\omega\frac{L}{W}})$ to one period. Here we choose to restrict ω to $[-\pi W/L, \pi W/L)$ to obtain

$$e^{-j\frac{c_i}{W}\omega}Y_i(e^{j\omega\frac{L}{W}})\mathbf{1}_{[-\frac{\pi W}{L},\frac{\pi W}{L})} = \frac{W}{L}\sum_{m=-\left\lfloor\frac{1}{2}(L+1)\right\rfloor+1}^{\left\lfloor\frac{1}{2}(L+1)\right\rfloor} e^{-j\frac{2\pi}{L}c_i m}\, X(\omega - 2\pi\tfrac{W}{L}m)\mathbf{1}_{[-\frac{\pi W}{L},\frac{\pi W}{L})},$$

for $i = 1, \ldots, q$, where $\mathbf{1}_{[\cdot)}$ denotes the indicator function. Note that the restriction to $[-\pi W/L, \pi W/L)$ removes the dependence on ω in the summation limits since within this interval $Y_i(e^{j\omega\frac{L}{W}})$ is a linear combination of a particular (finite) set of spectral segments of $x(t)$. We can therefore write this expression in a matrix-vector formulation

$$\mathbf{z}(\omega) = \mathbf{\Phi}\mathbf{s}(\omega) \tag{1}$$

where

$$z_i(\omega) = e^{-j\frac{c_q}{W}\omega}Y_i(e^{j\omega\frac{L}{W}})\mathbf{1}_{[-\frac{\pi W}{L},\frac{\pi W}{L})} \tag{2}$$

$$\mathbf{\Phi}_{i,l} = \frac{W}{L}e^{-j\frac{2\pi}{L}c_i m_l} \tag{3}$$

$$s_l(\omega) = X(\omega - 2\pi\tfrac{W}{L}m_l)\mathbf{1}_{[-\frac{\pi W}{L},\frac{\pi W}{L})} \tag{4}$$

for $i = 1, \ldots, q$, $l = 1, \ldots, L$, and $m_l = -\left\lfloor\frac{1}{2}(L+1)\right\rfloor + l$.

In MC sampling, (1) serves as the base equation for the reconstruction of $x(t)$. Support recovery, however, relies on forming the covariance matrix of $\mathbf{z}(\omega)$. As shown in the next section, our primary interest in the covariance matrix is not in using it to recover the support of $\mathbf{s}(\omega)$, but rather using it to estimate the PSD of $x(t)$.

3. MULTI-COSET PSD ESTIMATION OF WSS SIGNALS

Key ideas. Let $x(t)$ be a complex WSS process with PSD $P_x(\omega)$ and let $\mathbf{R_z}$ denote the integrated covariance matrix[2] of $\mathbf{z}(\omega)$,

$$\mathbf{R_z} \triangleq \int_{-\pi\frac{W}{L}}^{\pi\frac{W}{L}} \mathbb{E}[\mathbf{z}(\omega)\mathbf{z}^H(\omega)]\, d\omega \tag{5}$$

$$= \mathbf{\Phi}\left[\int_{-\pi\frac{W}{L}}^{\pi\frac{W}{L}} \mathbb{E}[\mathbf{s}(\omega)\mathbf{s}^H(\omega)]\, d\omega\right]\mathbf{\Phi}^H \tag{6}$$

$$= \mathbf{\Phi}\,\mathbf{R_s}\,\mathbf{\Phi}^H, \tag{7}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator, H denotes Hermitian transpose and $\mathbf{R_s}$ is the integrated covariance matrix of the spectral segments in $\mathbf{s}(\omega)$. We begin by recalling the fact that the Fourier transform (appropriately defined) of a WSS random process $x(t)$ is itself a nonstationary white noise process (in frequency) with covariance function [6, p. 418],

$$R_X(\omega,\nu) = \mathbb{E}[X(\omega)X^*(\nu)] = 2\pi P_x(\omega)\delta(\omega-\nu), \tag{8}$$

where * denotes complex conjugacy. This fact, along with (4), imply that the integrand in (6) equals

$$\mathbb{E}[s_k(\omega)s_l^*(\omega)] = 2\pi P_x^{(l)}(\omega)\delta_{k,l}, \tag{9}$$

where $\delta_{k,l}$ denotes the Kronecker delta function and $P_x^{(l)}(\omega)$ represents the PSD of the random process

$$s_l(t) = x(t)e^{j2\pi\frac{W}{L}m_l t} \star \tfrac{W}{L}\mathrm{sinc}(\tfrac{\pi W}{L}t),$$

[2] Unlike the formulations in [1], we introduce expectations here because the signals are random processes.

i.e., $P_x^{(l)}(\omega)$ is the PSD of a shifted, ideally low pass filtered version of $x(t)$. To understand (9), consider the case when $k = l$ and $\omega \in [-\pi W/L, \pi W/L)$. Then

$$\mathbb{E}[|s_l(\omega)|^2] = \mathbb{E}\Big[\big|X(\omega - 2\pi \tfrac{W}{L} m_l)\big|^2\Big] \\ = 2\pi P_x^{(l)}(\omega).$$

When $k \neq l$, it follows directly from (8) that $\mathbb{E}[\mathbf{s}_k(\omega)\mathbf{s}_l^*(\omega)] = 0$. Thus $\mathbf{R_s}$ is a matrix whose main diagonal elements represent the power in the L spectral segments of $x(t)$,

$$\Big[-\pi W + (l-1)\frac{2\pi W}{L}, -\pi W + l\frac{2\pi W}{L}\Big), \quad l = 1, \ldots, L$$

and whose off-diagonal elements are zero. Collectively, therefore, the diagonal elements of $\mathbf{R_s}$ form a finite resolution approximation to $P_x(\omega)$. This fact is the basis of the estimator introduced below: We estimate $\mathbf{R_z}$ from a finite set of MC samples and invert (7) to obtain estimates of the total power within these spectral segments. We first, however, further consider the structure of $\mathbf{R_s}$.

Exploiting covariance structure. The diagonal nature of $\mathbf{R_s}$ has significant implications for how we should solve (7). With the new indexing $k = q(a-1) + b$ ($1 \leq a, b \leq q$), let us rewrite (7) as

$$\mathbf{u}_k \triangleq [\mathbf{R_z}]_{a,b} = \sum_{l=1}^{L} \mathbf{\Phi}_{a,l}\mathbf{\Phi}_{b,l}^*[\mathbf{R_s}]_{l,l} \\ = (W/L)^2 \sum_{l=1}^{L} e^{-j\frac{2\pi}{L}(c_a - c_b)m_l}[\mathbf{R_s}]_{l,l} \qquad (10)$$

where again $m_l = \lceil -\frac{1}{2}(L+1)\rceil + l$. We can write (10) concisely in matrix-vector form,

$$\mathbf{u} = \mathbf{\Psi}\mathbf{v}, \qquad (11)$$

where $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_{q(q-1)/2+1})$, $\mathbf{\Psi}_{k,l} = (\frac{W}{L})^2 e^{-j\frac{2\pi}{L}(c_a - c_b)m_l}$, and $\mathbf{v} = \operatorname{diag}(\mathbf{R_s})$ with $k = 1, \ldots, \frac{q(q-1)}{2} + 1$ (q odd) and $l = 1, \ldots, L$. To determine $\mathbf{R_s}$ from $\mathbf{R_z}$ and hence infer $P_x^{(l)}(\omega)$, we need to solve (11). This requires that $\operatorname{Rank}(\mathbf{\Psi}) \geq L$. The maximum number of distinct differences $c_a - c_b$ equals $\frac{q(q-1)}{2} + 1$, therefore we know $\operatorname{Rank}(\mathbf{\Psi}) \leq \frac{q(q-1)}{2} + 1$.

As we are assuming that $x(t)$ is complex while we know that $\mathbf{v}$ must be real we can double the number of equations by solving:

$$\begin{bmatrix} Re(\mathbf{u}) \\ Im(\mathbf{u}) \end{bmatrix} = \begin{bmatrix} Re(\mathbf{\Psi}) \\ Im(\mathbf{\Psi}) \end{bmatrix} \mathbf{v}. \qquad (12)$$

Suppose the MC sampling pattern $\{c_i\}$ is chosen such that

$$\operatorname{Rank}\left(\begin{bmatrix} Re(\mathbf{\Psi}) \\ Im(\mathbf{\Psi}) \end{bmatrix}\right) = q(q-1) + 2. \qquad (13)$$

Then as long as q and L satisfy $q(q-1)+2 \geq L$, we can invert (12) using the pseudoinverse of Ψ and retrieve $\mathbf{R_s}$ from $\mathbf{R_z}$. If (13) holds as L grows, it is clear that we can retrieve a finite resolution approximation to $P_x(\omega)$ with any sampling rate, given appropriate q values. If either L or q is held fixed while the other is allowed to vary, the above inequality establishes a minimum average sampling rate that is required to successfully obtain a PSD approximation.

Exploiting covariance structure and signal sparsity. Suppose now $P_x(\omega)$ is the PSD of a sparse multiband signal. As such, $\mathbf{v}$ admits a sparse representation for which we can solve even when $q(q-1) + 2 < L$, provided certain conditions on $\mathbf{\Psi}$ and the size of the spectral support are satisfied. We do not pursue the recovery conditions here; we instead focus on the derivation of a CS style spectral approximation in this setting.

A sparse representation for $\mathbf{v}$ can in theory be computed by solving the ℓ_0 minimization problem, $\widehat{\mathbf{v}} = \operatorname{argmin}\|\mathbf{v}\|_0$ subject to (12). It is thus tempting, following standard compressed sensing practice, to replace it with a simpler ℓ_1 minimization problem. This step is unnecessary, however, because (12) has additional structure that simplifies the problem.

First, observe that the unknown power spectrum $\mathbf{v}$ is by definition nonnegative. Second, recall that the rows of Ψ enumerate the possible differences $c_a - c_b$ and that the row representing the difference $c_a - c_b = 0$ is a row of ones. The first point means that we are fundamentally interested in solving a nonnegative sparse representation problem. The second point serendipitously allows us to compute a sparse solution using nonnegative least squares without explicity imposing a sparsity constraint [11]. The reason is that the row of ones measures the ℓ_1 norm of $\mathbf{v}$, and thus a nonnegative least squares solution automatically promotes a sparse solution—we in essence get the a sparse solution for free.

PSD estimation. We now want to estimate the multiscale approximation to $P_x(\omega)$ outlined above using a finite set of MC samples $\{y_i(k)\}$, $k = 0, \ldots, N-1$, $i = 1, \ldots, q$. To do so, we re-express (5) in terms of the samples. By applying the discrete-time Fourier transform's orthogonality property [10, pp. 90-91], it can be shown that

$$[\mathbf{R_z}]_{l,m} = 2\pi\frac{W}{L}\sum_{k=-\infty}^{\infty}\mathbb{E}\Big[y_l(k - \frac{c_l}{L})y_m^*(k - \frac{c_m}{L})\Big],$$

where the notation $y_l(k - \frac{c_l}{L})$ denotes a fractional shift of the sequence $y_l(k)$. With a finite number of samples, we estimate $\mathbf{R_z}$ as

$$[\widehat{\mathbf{R}}_\mathbf{z}]_{l,m} = 2\pi\frac{W}{L}\sum_{k=0}^{N-1} y_l(k - \frac{c_l}{L})y_m^*(k - \frac{c_m}{L}), \qquad (14)$$

where here the fractional delays can be computed by a fractional delay filter at baseband frequencies. $\widehat{\mathbf{R}}_\mathbf{z}$ is then used with the methods described above to invert (12) to obtain the finite resolution PSD estimates $\widehat{\mathbf{v}}$. This two step process comprises our proposed PSD estimator. Like DASP, this estimator can yield PSD estimates of arbitrary resolution (scale) and at arbitrarily low sampling rates.

There are two sources of error for this estimator. The first derives from estimating $\mathbf{R_z}$ from a finite amount of data. The second derives from the approximate least squares solution. It is straightward to show that $\widehat{\mathbf{R}}_\mathbf{z}$ is an asympotically unbiased estimator of $\mathbf{R_z}$; thus as N increases the PSD estimate becomes more accurate on average (provided (12) can be inverted).

4. NUMERICAL EXAMPLES

Example 1. Let $x(t)$ be white Gaussian noise bandlimited to 500 Hz with two 40 Hz stop bands ($W = 1000$ Hz). Consider a 25 channel MC sampler where each channel samples $x(t)$ at a rate of 10 Hz over a 2000 sec window ($q = 25, L = 100$, average sampling rate of 250 Hz). In this scenario, the channels sample at a rate that is ten times smaller than the Nyquist rate with the resolution of the estimator being $W/L = 10$ Hz. Because $q(q-1) + 2 \geq L$ in this example, we can compute the estimator regardless of the spectral sparsity of $P_x(\omega)$ and can invert (12) using either the pseudoinverse or a nonnegative least squares approach. Figure 2 illustrates

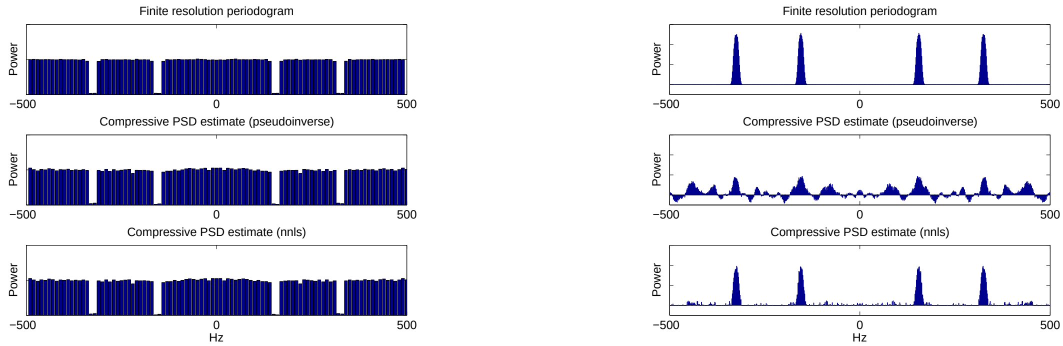

Fig. 2. The plots show two compressive PSD estimates in comparison to periodograms (top panels) computed from a realization of the original (simulated) random process $x(t)$. Here, the periodograms can be considered to be the true mulitscale approximations that the compressive estimates are estimating. The left hand plots derive from a non-sparse PSD with spectral holes; the right hand plots derive from a sparse multiband signal.

these facts. One could identify this example as a rudimentary cognitive radio scenario, where the task is to identify spectral "holes" to exploit. Figure 2 shows two finite resolution periodograms computed from the original (simulated) input random process $x(t)$. For our purposes, these periodograms can be considered to be the "true" representations of the finite resolution PSD approximations the compressive PSD estimates are estimating. Comparing the compressive PSD estimates to these periodograms shows that the compressive estimates may be more than sufficient to detect spectral holes.

Example 2. The second example is a variation of the first where we decrease the MC sampling rate by increasing L such that $q(q-1)+2 < L$ ($L = 800$), and we change $x(t)$ to be a sparse multiband signal. The resultant channel sampling rate is approximately 1.5 Hz with a system average sampling rate of about 36 Hz. (This rate is about half the minimal Landau rate of 80 Hz.) As expected, Figure 2 shows that the pseudoinverse fails to invert (12) because $q(q-1)+2 < L$, but the nonnegative least squares estimator produces an estimate that can accurately identify the active bands. We stress that in this case the nonnegative least squares derived estimate continues to improve as more samples are taken, but the errors in the pseudoinverse derived estimate are systematic and remain irrespective of the number of samples.

5. CONCLUSION

In this paper, we presented a preliminary development of a CS based estimator that estimates the PSDs of WSS processes at multiple resolutions (scales) and at arbitrarily low sampling rates. The approach relies on the spectral characteristics of WSS processes, MC sampling, and the particular form of the linear system of equations that describe multi-coset sampling. The estimator applies to both spectral sparse and non-sparse PSDs, yet can take advantage of CS techniques when the true PSD is sparse. The method also interrelates the fields of CS and DASP by showing that the multi-coset sampler can be used to perform a DASP-type spectral estimation. The examples demonstrate the potential effectiveness of the method, but a more thorough analysis of the estimator's consistency is needed.

Lastly, this estimation technique can, in principle, be extended to other CS sampling structures. In particular, the modulated wideband converter [4] operates on the same principles as the MC sampler and results in nearly the same Fourier description. Thus, despite the differences in the systems' sampling mechanisms, it should be possible to develop a similar estimator for the modulated wideband converter.

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